OTS: 60-11,920

JPRS: 5026

14 July 1960

NET THEORY ELEMENTS AND THEIR APPLICATION IN NOMOGRAPHY

- USSR -

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NET THEORY ELEMENTS AND THEIR APPLICATION IN NOMOGRAPHY

[This is a translation of an article written by T. Steyskalova in Vychislitel'naya Matematika (Computational Mathematics), No. 4, 1959, pages 173-183.]

Lately, a number of individuals engaged in nomography have shown noticeably greater interest in work on topological net theory. Considerable attention was devoted to these problems by the Hamburg Seminar, under the guidance of Blaschke.

The results obtained in the past thirty years may be seen as forming a self-contained, formal discipline, having its own methods and problems.

Far from everything dealt with by this discipline has direct relation to nomography. Yet, acquaintance with certain of the results of net theory must be considered absolutely essential to nomographers.

The paper that follows presents some of the results contained in Blaschke's book [1] which have bearing on nomography. In addition, we adduce here a theorem, which parallels the Rademeister theorem, the formulation and proof of which were kindly supplied to us for this article by G. Ye. James-Levi.

1. Basic Concepts of Net Theory

Let us be given families of curves δ1, δ2, δ3 in an area G on a plane (x, y).

$$u_i(x,y) = \text{const}, \; i=1, 2, 3.$$

where u_i are analytic functions within G.

$$\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2 \neq 0 \quad \text{at each point of area G;}$$

one and only one curve of each family passes through each point in G; Jacobian:

$$\left|\frac{\partial (u_j, u_k)}{\partial (x, y)}\right| \neq 0, \; j, k = 1, 2, 3, \; j \neq k;$$

two curves of different families have no more than one point in common.

Let us call such a system of curves a <u>triple system</u>. It must be assumed that each curve of a triple system has a single continuous arc in common with area G.

The simplest triple system is a regular net, i.e. a system consisting of three families of parallel straight lines, where lines of different families form angles of 60°. It is apparent that such a net forms regular hexagons, whose sides and diagonals are the straight lines of the regular net. Let us call such hexagons Brianchon hexagons, or simple B figures.

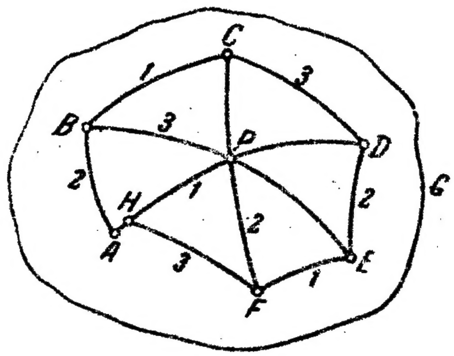

Fig. 1. Brianchon Figure

We will call B figures all figures constructed in the following manner (Fig. 1). Let a triple system be given in area G. We select any point P within G and plot through it lines of all three families, which we designate as 1_p, 2_p, 3_p. Here, the numeral designates the family to which the line belongs, while the sub-index indicates the point through which it passes. On line 1_p we select point A, which is sufficiently close to point P (i.e. such that the entire figure plotted by us is situated within G), and we plot line 2_A until it intersects 3_p at point B. Then we plot line 1_B until it intersects 2_p at point G. In a similar manner, we obtain points $D \equiv (3_C \times 1_P)$, $E \equiv (2_D \times 3_P)$, $F \equiv (1_E \times 2_P)$ & $H \equiv (3_F \times 1_P)$.

Point H may coincide with A and thus close our B figure, though this is not obligatory.

Triple systems in which all B figures are closed may be termed hexagonal. It is apparent that triple systems obtainable from a regular net by topological projection will be hexagonal.

2. Auxiliary Propositions

Let us be given a triple system within a certain area G. Let us examine the curvilinear triangle ABC, whose sides are formed of curves of the 1st, 2nd and 3rd families, and which pertains entirely to G. We may call such a triangle a coordinate

triangle. We select four points on side AB: P, Q, R, S (Fig. 2). We plot lines of the 2nd family through points P, R, and lines of the third family through points Q, S.

The intersection points $V \equiv (2_P \times 3_Q)$ and $W \equiv (2_R \times 3_S)$ are situated within our triangle. Indeed, let us assume, for example, that point V lies outside Δ ABC. Then curve 2_p will intersect side AC (or curve 3_Q will intersect side CB), and this is impossible according to the definition of a triple system (a single curve of each family passes through each point $\in G$). It may be shown similarly that W is likewise situated within triangle ABC.

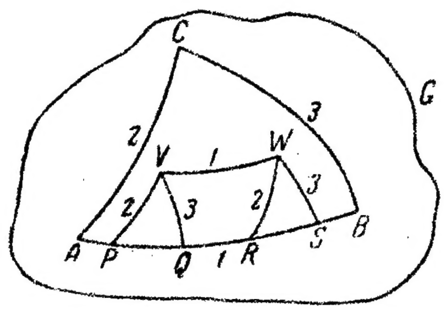

Fig. 2. Coordinate Triangle

Let us plot lines of the first family l_V and l_W through points V and W and introduce the following definition: arcs ("segments") PQ and RS may be called equal, when points V and W are situated on the same line of the 1st family, i.e. when $l_V \equiv l_W$. From this definition, there follow the properties of reversibility and transitiveness of the equation, i.e. 1) when PQ = RS, then RS = PQ; 2) when PQ = RS and RS = TU, then PG = TU.

Let us now introduce a definition of the inequality of the "segments". Let us say that PQ < RS when line l_V lies between curves l_W and AB. This definition is also transitive and possesses the following property: if points R, S are situated on arc PQ, then RS $\leqq$ PQ, and become equal only when $P \equiv R$, $S \equiv Q$. This property follows from the fact that point W always lies within or on the edge of triangle PQV and therefore that line l_V is situated between l_W and AB and that $l_V \equiv l_W$ only when $V \equiv W$ i.e. $P \equiv R$ and $S \equiv Q$.

Let us now return to our examination of coordinate triangle ABC. We select arbitrarily on side AB a point $P \equiv P_1$ and we plot point P_2 in such a manner that P_1P_2 = AP, point P_3 so that P_2P_3 = AP, etc. (we understand equality here and below

as defined earlier). We thus obtain a sequence of points $\{P_i\}$.
This sequence is monotonous, i.e. if A is to the left of P, then P_i is also to the left $P_i + 1$ for any i (Fig. 3). Let us assume the opposite, i.e. let i be found to be such that all P_k's where k i are to the left of P_{k+1}, while P_i is to the right of P_{i+1}. Then $3_{P_{i+1}}$ (or 2_{P_i} and $2_{P_{i+1}}$) have a point in common, and this is possible.

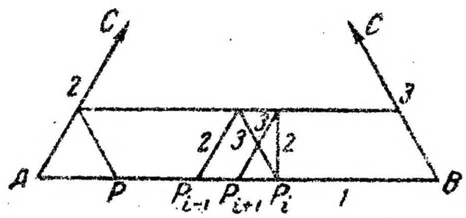

Fig. 3. Monotonous nature of sequence of P_i points

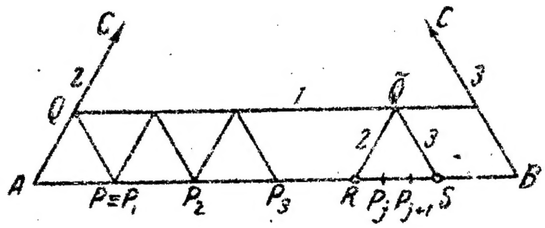

Fig. 4. Existence of limit point in $\{P_i\}$ sequence

Let us now show that point P_i will reach or pass beyond point B after a finite number of steps. Let us assume that this is not so. We would then have, on finite "segment" AB, an infinite monotonous sequence of points P_i. Therefore, this sequence must necessarily have at least one limit point S (Fig. 4). We plot point R in such a manner that RS = AP and that R be to the left of S. This may always be done by plotting 3_S until it intersects 1_Q at point $\tilde{Q}$, and plotting $2_{\tilde{Q}}$ until it intersects AB. R will be the point of intersection $(2_{\tilde{Q}} \times AB)$ However, since S is the limit point for P_i, then, anywhere near point S (i.e. on "segment" RS as well), there must exist an infinite number of points P_i. Let us take two such points: P_j and $P_{j\ 1}$, so that they lie strictly within RS. Then, however, as shown earlier, $P_jP_{j+1} < RS = AP$, and this is

contradictory to the construction of "segment" P_jP_j+1. Thus, the natural number n will always be found to be such that or that P_n is to the right of B.

Let us demonstrate, furthermore, that it is always possible to plot point P in such a manner than $P_n \equiv B$ for any predetermined natural number n. In other words, it is always possible to divide AB into n equal "segments". For this purpose, let us first demonstrate that if P, Q are two points on AB and P is to the left of Q, then, by plotting point sequences $\{P_i\}$ and $\{Q_i\}$ as earlier described, for any iP_i is to the left of Q_i if only Q_i is still on AB. Let this be not so, i.e. let all P_j's where $j<k$, be situated to the left of Q_j, while P_k is to the right of Q_k. Then, there are at least two Q_i points between P_{k-1} and P_k, namely Q_{k-1} and Q_k. However, as we have shown earlier, in that case $Q_{k-1}Q_k < P_{k-1}P_k$. i.e. we arrive at a contradiction, since, according to the plotting of points P_i and Q_i

$$Q_{k-1}Q_k = AQ > AP = P_{k-1}P_k.$$

Let us now be given a natural number N. Let us show how to plot initial point P so that $P_N \equiv B$. . We take any point P^1 on AB and plot the sequence $\{P_i^1\}$. If $P_N^1 \equiv B$, then P^1 is sought point P. If P_N^1 is to the right of B, we then take point P^2 to the left of P^1 on AB, and, if P^1 is to the left of B, we take point P^2 to the right of P^1. We then plot the sequence $\{P_i^2\}$ points and repeat our reasoning. We thus obtain a convergent point sequence $\{P^k\}$. The limit point of this sequence will be sought point P.

Let us now examine a specific case, that of a regular net. Here, the curvilinear coordinate triangle ABC is a regular (= equilateral) triangle, and the definition of the equality of the curvilinear segments coincides with the usual understanding of the equality of segments. We take coordinate triangle ABC, divide all of its sides into n parts (n being a natural number), and plot lines of all families through the points of division. We obtain the figure shown in Fig. 5 which we will refer to from now on as the <u>D_n figure</u>. We will similarly designate any topological transformation of this figure.

<u>Lemma 1</u>. A D_n figure may be constructed within any hexagonal net.

<u>Proof</u>. We may provide proof for this assertion by the method of mathematical induction. Let n = 3 to begin with, since the net under consideration is hexagonal, and the hexagon of straight net lines is to be closed.

Let us admit that our proposition holds true for n - 1, and let us show that it will then also hold true for n. Let us divide AB into n parts by means of points P_i. Through these points we plot lines of the 2nd and 3rd families. Points Q_i are located on a single line of the 1st family, since all $P_i P_{i+1}$ segments are equal to one another. However, all R_i points are located on a single line of the 1st family (Fig. 6), since, in the net under consideration, the hexagons with centers at points Q_i must close, i.e. each pair of points $R_i R_{i+1}$ (and, therefore, all these points) is situated on a single line of the 1st family. It follows that the distances between two neighboring Q_i points are equal, i.e. that triangle $Q_o Q_{n-1} C$ is a D_{n-1} figure. According to the supposition of the induction, the lemma holds true for n - 1. Thus we have shown that a D_n figure may be constructed in any hexagonal net.

Lemma 2. The definition of the equality of curvilinear segments in hexagonal nets also possesses the property of additiveness.

Proof. It is apparent from a D_n figure (Fig. 6), that not only segments of the form $P_i P_{i+1}$, but also all segments of the form $P_i P_{i+k}$ are equal, k being a natural number (for example, for segments $P_i P_{i+2}$, equality follows from the fact that the R_i points are situated on one line of the 1st family).

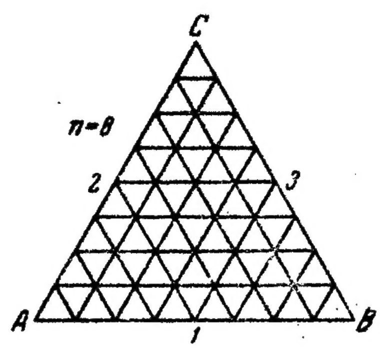

Fig. 5. D_n figure

Let us be given four "segments" MN, NP, RS and ST on side AB of the coordinate triangle ABC, such that MN = RS and NP = ST. Let us demonstrate the additive nature of the equality, i.e. that MP = RT (Fig. 7).

Let points M, N, P, R, S, T initially be the points of division of certain D_n figures. We will designate a D_n figure containing point M among its points of division as figure D_n, etc. Let us take the D_n figure containing figures

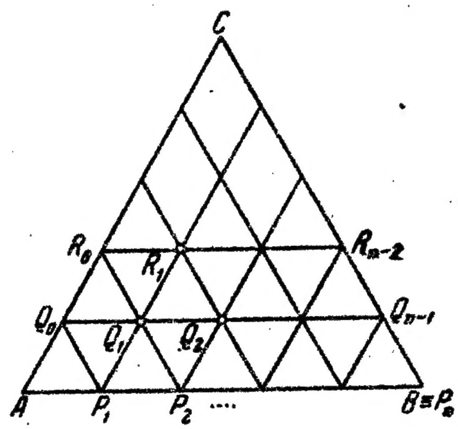

Fig. 6. Construction of D_n figure in hexagonal net, assuming the existence of a D_{n-1} figure.

D_{n_M}, D_{n_N}, D_{n_P}, D_{n_R}, D_{n_S}, D_{n_T} (D_n will be such a figure, for example, when $n = n_M \cdot n_N \cdot n_P n_R \cdot n_S \cdot n_T$). As noted earlier, additiveness will hold true for this figure. If now some (or all) of the points M, N, P, R, S, T are not points of division for any D_n figures, then additiveness follows from the fact that any such point may be included within an interval as minute as desired, whose end-points are points of division of certain D_n figures.

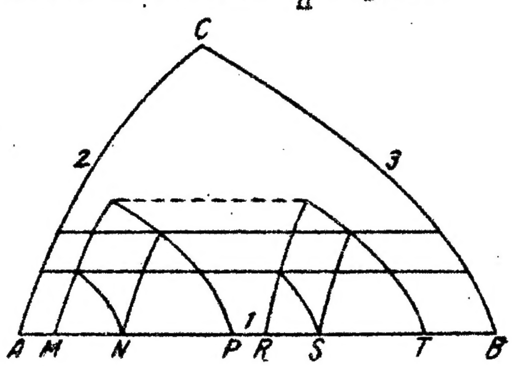

Fig. 7. Additive nature of segment equality.

Remark. If D_m and D_n figures are such that $m = k \cdot n$, where k is a natural number, it is apparent all lines of figure D_n are contained within figure D_m.

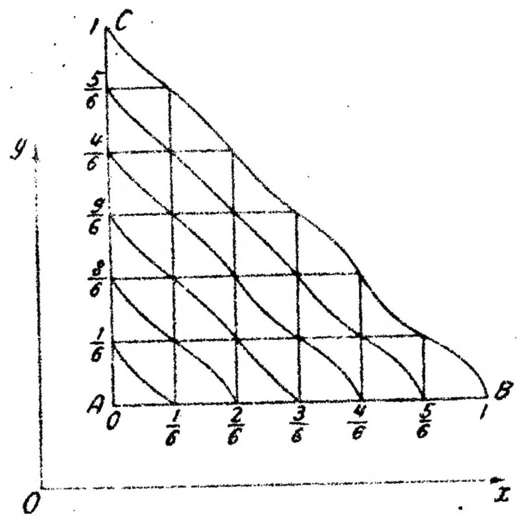

Fig. 8. D_n figure in net where two line families are parallel to coordinate axes Ox and Oy.

3. Basic Theorem of Hexagonal Nets

Theorem. A given triple system may be topologically transformed into a regular net when, and only when all Brianchon B figures are closed (i.e. when this curvilinear net is hexagonal).

Proof. To demonstrate the necessity of this, let us assume that the B figure will not close, i.e. that $A \not\equiv H$. However, then the images of these points in topological projection will not coincide, and this will contradict the definition of a regular system.

Now let us demonstrate the sufficiency of the theorem.

Let us be given a hexagonal net. Let us project the 1st and 2nd families of coordinate lines on straight lines parallel to the Ox and Oy axes of a Cartesian coordinate system. This projection will be topological (according to the conditions imposed on the lines of the triple system). Let us examine the coordinate triangle ABC and inscribe within it all possible D_n figures (Fig. 8).

On segment AB let us plot function $f_1(x)$ so that the p-th point of division of figure D_q receives a notation of $\frac{p}{q}$. Let us show that this notation is the only one. Let point P be the p-th point of division of figure D_q and the r-th point of division of figure D_s. Let us posit that $s > q$. Then $s = kq$ (k is a whole number), i.e. figure D_q is contained within D_s. But then $r = kp$ also, and therefore $\frac{p}{q} = \frac{r}{s}$ i.e. there

corresponds to each point of division of segment AB a single notation. Then point A will evidently have a notation of 0, and point B a notation of 1. We have thus established the mutually univalent relation which holds between the points of division of segment AB and rational numbers on the segment [-0.1] of the numerical right line. We will term the points of division rational points, and other points, irrational. It is evident that function $f_1(x)$ is monotonous.

We still need to fully define $f_1(x)$ for irrational points of segment AB. Let R be an irrational point. It then divides all rational points into two classes, one of which is situated to the left of R, and the second of which is to the right of R. The notation of this point must divide the rational numbers into two classes such that all rational numbers in the second class are larger than the numbers in the first class. Since the points of division of D_n figures are distributed continuously, it follows that the notation of the irrational point is a Dedekind section, i.e. an irrational number. Thus, function $f_1(x)$ defined will be continuous and monotonous. In analogous fashion, we define the continuous monotonous function $f_2(y)$ on AC.

Let us now show that the following relation holds for any curve of the 3rd family:

$$f_1(x) + f_2(y) = \text{const.} \tag{1}$$

Let us first examine a line entering into D_n. For all the points of this line that are intersection points of a certain D_n figure, relation (1) obviously holds. However, the line in question also enters into D_{2n}, D_{3n}, ... D_{kn}, i.e. relation (1) holds for the intersection points of these figures as well. However, since these points are distributed continuously over the entire line in question and the line itself is continuous, (1) holds also for any point of that curve.

For a line of the 3rd family, passing through an irrational point on segment AB, the values of sum $f_1(x) + f_2(y)$ are enclosed between two rational values, which may be arbitrarily brought closer together. Therefore for such a line as well, relation (1) holds true.

Now, let us effect the transformation

$$X = f_1(x), \quad Y = f_2(y). \tag{2}$$

This transformation will be topological, since functions $f_1(x)$ and $f_2(y)$ are continuous and monotonous. In this transformation, the first two families of coordinate lines will evidently remain on straight lines parallel to axes Ox and Oy, while the third family, for which we have x + y = const., will be

transferred to the family of straight lines that are parallel to the bisectrix of the second coordinate angle. By an affinal transformation, it is now possible to convert these three groups of parallel straight lines into three groups of parallel straight lines forming angles of $\frac{\pi}{3}$ in pairs.

Thus, the theorem is fully demonstrated for any coordinate triangle situated within area G, where the net under examination is given to us. It follows for the entire area G, since G may be covered with such coordinate triangles, inasmuch as coordinate lines are everywhere continuously distributed.

4. Thompson and Rademeister Figures

Along with Brianchon's B figure, let us examine Thompson's T figure and Rademeister's R figure (Fig. 9).

Theorem 2. To have a triple system be a hexagonal net, it is necessary and sufficient that any T (or R) figure in this system be closed.

Proof. Let us first provide the demonstration for a T figure.

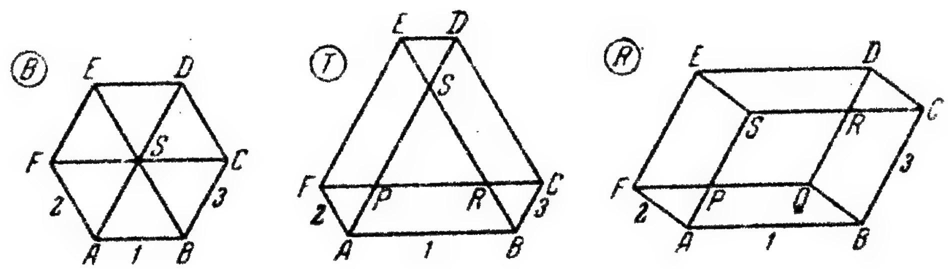

Fig. 9. Brianchon, Thompson and Rademeister figures

The sufficiency of the theorem is obvious, since the B figure is a particular case of a T figure, in which points P and R coincide. And since all T figures are closed, all B figures are closed also, i.e. our net is hexagonal.

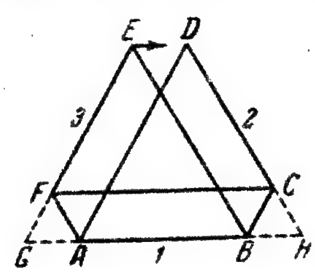

Fig. 10. Closure of T figure on assumption that B fiugres close

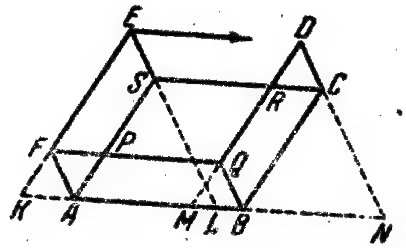

Fig. 11. Closure of R figure on condition that all B figures close

Its necessity follows from the property of additiveness of hexagonal nets. Indeed, we wish to prove that points E and D (Fig. 10) are on the same line of the 1st family, i.e. $l_E \equiv l_D$. Let us prolong EF, CD and AB until they intersect at points G and H, and let us examine "segments" GB and AH. These "Segments" are equal, because GB = GA + AB, AH = AB + BH and GA = BH, and therefore points F and 6 are situated on the same line of the 1st family. But since GB = AH, according to the definition of the equality of curvilinear segments, points E and D are situated on the same line of the 1st family, Q. E. D.

The demonstration proceeds in an analogous manner for an R figure. The sufficiency of the theorem follows from the fact that a B figure is a particular case of an R figure, in which points P, Q and S coincide. To demonstrate its necessity, we make use again of the property of additiveness of hexagonal nets, and we shall prove that $l_E \equiv l_D$ (Fig. 11). We prolong lines AB, EF, ES, DQ and DC and we designate the points of intersection as KL = MN. Indeed, KL = KA + AL, MN = MB + BN. But KA = MB, since $l_F \equiv l_Q$, and AL = BN, since $l_S \equiv l_C$. Thus, in view of additiveness, we have KL = MN. The theorem has been fully demonstrated,

5. The Application of Net Theory to Nomography

We have demonstrated that any triple system in which all T figures (or all R figures, or all B figures) are closed may be projected topologically on a regular net. This is of great importance for nomography. Indeed, if we plot a cartesian abacus for a given equation, and its lines intersect at very acute angles, large errors are possible in the use of the nomograph.

For this reason, it is desirable that the angles of intersection of lines in the abacus be as large as possible. The ideal case is that of intersection at angles of 60°. This case takes place precisely when the abacus forms a regular net. It is therefore important to find the equations which

always allow the construction of such an abacus. Third order nomographic equations are equations of this kind.

Theorem 3. In order that an equation

$$F(x, y, z) = 0 \tag{3}$$

be a third order nomographic equation, it is necessary and sufficient that of the six equations

$$\left.\begin{array}{lll} F(x_0, y_1, z_1) = 0 & F(x_0, y_2, z_2) = 0 & F(x_1, y_2, z_3) = 0 \\ F(x_2, y_1, z_3) = 0 & F(x_2, y_0, z_2) = 0 & F(x_1, y_0, z_1) = 0 \end{array}\right\} \tag{4}$$

each one be the consequence of the other five.

Let us clarify the formulation of this theorem. Let us assume that a cartesian abacus has been plotted for equation (3). We take three arbitrary lines of level z in that equation: $z = z_1$, $z = z_2$, $z = z_3$. Each one of the equations (4) $F(x_i, y_j, z_k) = 0$ means that lines $x = x_i$, $y = y_j$, $z = z_k$ intersect at a single point, i.e. that the Thompson figure obtained (Fig. 12) is closed. But since all figures are closed, this means that the cartesian abacus forms a hexagonal net. Thus, having demonstrated this theorem, we show that it is possible to plot a cartesian abacus whose lines form a regular net for any 3rd order nomographic equation.

Proof. Let us first demonstrate the sufficiency of the theorem.

It is posited that the level lines of equation (3) form a hexagonal net. It is required to demonstrate that (3) is a 3rd order nomographic equation. In demonstrating the basic theorem of hexagonal nets, we found that the relation $X + Y = \text{const.}$ holds true for every line of the 3rd family, Z and Y being the notations of lines of the first two families. Taking now this constant as the notation of the corresponding line of the 3rd family and designating it as Z, we get

$$X + Y = Z$$

This is an equation of the 3rd nomographic order.

Now let us demonstrate the necessity of the theorem. Let us be given a 3rd order nomographic equation. We must prove that each one of the equations (4) is a consequence of the five others. We know that along with a Cartesian abacus, it is possible to construct a nomograph on a conical section and a right line for a 3rd order nomographic equation. There exists a mutually-univalent relation between the nomograph and the cartesian abacus, and this relation is double (Fig. 13).

The equation $F(x_i, y_j, z_k) = 0$ means, on the abacus, that level lines $x = x_i$, $y = y_j$, $z = z_k$ pass through a single

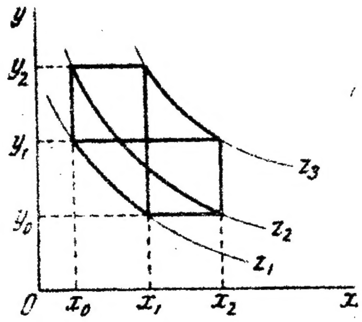

Fig. 12. Thompson figure in cartesian abacus for 3rd order nomographic equation.

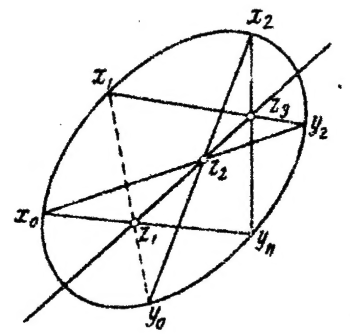

Fig. 13. Figure, corresponding to a Thompson figure on nomograph of equalized points.

point. On the nomograph, this corresponds to the fact that points x_i, y_i, z_k are on the same straight line. Let us assume, for example, that the first five equalities in (4) are satisfied, and let us demonstrate the sixth. We take points x_o, x_1, x_2, y_o, y_1, y_2, z_1, z_2 and z_3, satisfying the first five equations in (4).

Let us plot the straight line (x_1, y_o) and show that it passes through point z_1 (Fig. 13), i.e. that $f(x_1, y_o, z_1), = 0$. Indeed, straight lines (x_o, y_1, z_1), (x_o, y_2, z_2), (x_1, y_1, z_2) and (x_2, y_o, z_2), together with straight line (x_1, y_o), form a Pascal configuration with apexes on the two x and y scales, situated on a 2nd order curve, and the Pascal right line belong to the third scale z. But then, according to the Pascal theorem, right line (x_1, y_o) must pass through z_1, Q.E.D.

Remark. In demonstrating the necessity of the theorem, we made use of results derived from projective geometry. However, the reverse is also possible, and to demonstrate certain theorems of projective geometry from the properties demonstrated above for hexagonal nets.

We have seen that the Thompson figure corresponds to the Pascal configuration. It is evident that there corresponds to the B figure that particular case of the Pascal configuration in which points x_1, y_1, z_2 are situated on a single right line (Fig. 14). Let us now find out what corresponds to an R figure. Let a quadrangle be plotted on a certain 2nd order curve K and a right line P as shown in Fig. 15. We will designate the corners of the quadrangle as x_1, y_1, x_2, y_2, and the points of intersection of the sides of quadrangle and straight line p as z_1, z_2, z_3, z_4. We then have the following theorem, which parallels the Rademeister theorem.

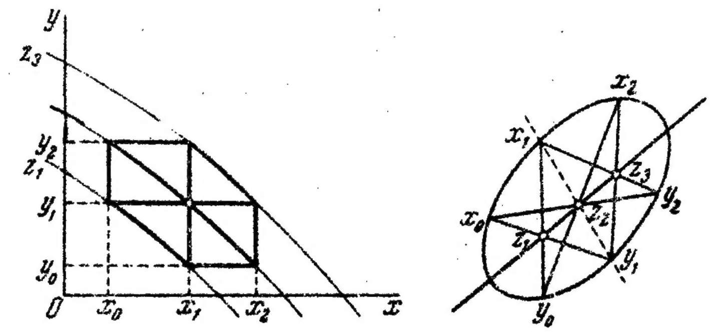

Fig. 14. Brianchon figure in cartesian abacus and its corresponding figure in a nomograph of equalized points.

Any quadrangle C, whose sides pass through points z_1, z_2, z_3, z_4 of right line p and whose apexes are situated on second-order curve K will be closed if one such quadrangle C_o is closed.

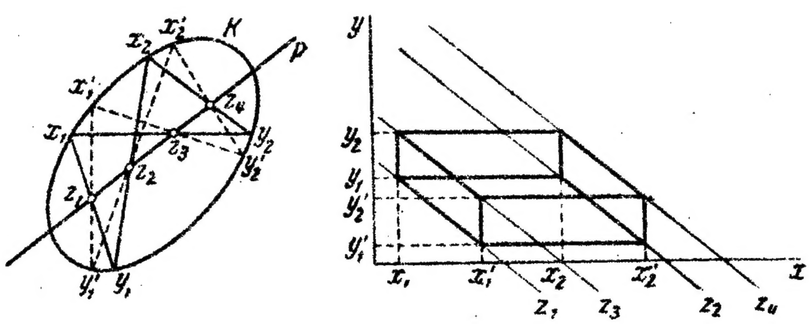

Fig. 15. Rademeister figure in cartesian abacus and corresponding figure in a nomograph of equalized points.

The proof of this theorem follows directly from the closure of the R figure, since there corresponds to every pair of quadrangles $C(x'_1, y'_1, x'_2, y'_2)$ and $C_o\,(x_1, y_1, x_2, y_2)$ an R figure in a hexagonal net.

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